BOUNDARY DUAL OF THE BULK SYMPLECTIC FORM

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work in progress
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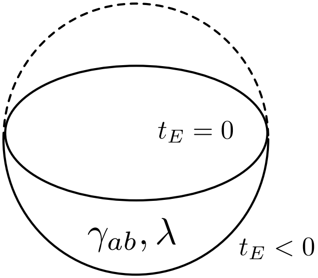
Motivation

Euclidean path integral states in AdS/CFT?

What are bulk coherent states $|h, K\rangle$? Can we understand their overlaps? Can we use wave functions $\langle h, K|\Psi\rangle$ to understand "how classical" the state $|\Psi\rangle$ is?

Candidate: Euclidean path integral states

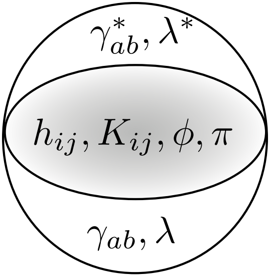
$$|\lambda\rangle = Pe^{\int_{t_E<0} \lambda(t_E,x)O(t_E,x)}|0\rangle$$



Motivation

Euclidean path integral states in AdS/CFT?

$\langle\lambda|\lambda\rangle =$



- Corresponding classical bulk: Euclidean saddle in the gravity calculation of $\langle\lambda|\lambda\rangle$ [Skenderis,van Rees]. Also leads to Lorentzian initial data. Need $\lambda(t_E \to 0) \to 0$ [Marolf,Parrikar,Rabideau,Rad,Raamsdonk]
- Around the vacuum, $|\lambda\rangle$ states are perturbative coherent states [Botta-Cantcheff,Martinez,Silva]
- The $|\lambda\rangle$ state is the vacuum of the Hamiltonian deformed by the coupling λ (if no dependence on t_E).

Today's plan:

Quantum Kähler structure on the space of complexified couplings (λ, λ^*)

$$\Updownarrow$$

Berry curvature for the parameters (λ, λ^*)

$$\Updownarrow$$

Gravitational symplectic form in the bulk

(In the classical limit, but for arbitrary large backgrounds)

If time allows: some preliminary results on the dual of the volume of a maximal slice

Quantum Kähler structure

Fubini-Study metric: The natural metric on the space of rays $\{|\psi\rangle \sim \mu|\psi\rangle | \mu \in \mathbb{C}\}$, induced by the inner product on $\mathcal{H}$.

$$ds^2 = \frac{\langle \delta\psi|\delta\psi\rangle}{\langle\psi|\psi\rangle} - \frac{|\langle\psi|\delta\psi\rangle|^2}{\langle\psi|\psi\rangle^2}$$

Now parametrize states as $(\alpha, \alpha^*) \mapsto |\alpha\rangle$ such that the conjugation comes from the inner product on $\mathcal{H}$:

$$\partial_{\alpha_i^*}|\alpha\rangle = 0, \quad \partial_{\alpha_i}\langle\alpha| = 0.$$

Pull-back of FS is automatically **Kähler**:

$$ds^2 = \partial_\alpha \partial_{\alpha^*} \log\langle\alpha|\alpha\rangle d\alpha d\alpha^*$$

Quantum Kähler structure

- **Kähler manifold comes with**
  - Complex structure J, or equivalently (anti)holomorphic coordinates (α, α^*)
  - A Kähler potential $\mathcal{K}(\alpha, \alpha^*)$, which determines

$$\text{the metric:} \quad ds^2 = \partial_\alpha \partial_{\alpha^*} \mathcal{K} d\alpha d\alpha^*$$

$$\text{the Kähler form:} \quad \omega = i\partial_\alpha \partial_{\alpha^*} \mathcal{K} d\alpha \wedge d\alpha^*$$

- So we also have closed **Kähler form** readily available:

$$\omega = i\partial_\alpha \partial_{\alpha^*} \log\langle\alpha|\alpha\rangle d\alpha \wedge d\alpha^*$$

- **Example:** For SHO coherent states:

$$|\alpha\rangle = e^{\alpha a^\dagger}|0\rangle, \quad \langle\alpha|\alpha\rangle = e^{\frac{|\alpha|^2}{2}}.$$

This results in the Kähler potential $\mathcal{K} = \frac{|\alpha|^2}{2}$ and symplectic form

$$\omega_\alpha = \frac{i}{2} d\alpha^* \wedge d\alpha.$$

Written in terms of $d\alpha = dq + idp$ this is

$$\omega_\alpha = dp \wedge dq.$$

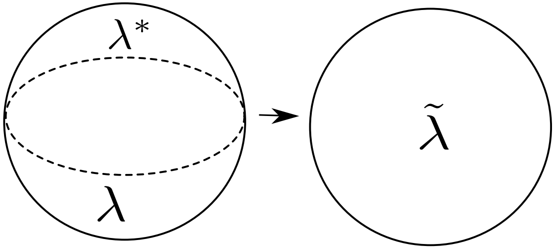
Quantum Kähler structure in QFT

This construction automatically gives a symplectic form on the space of Euclidean sources in any QFT.

$$|\lambda\rangle = e^{-\int_{t_E<0} dx \lambda(t_E,x) O(t_E,x)}|0\rangle, \quad \langle\lambda| = \langle 0|e^{-\int_{t_E>0} dx \lambda^*(-t_E,x) O^\dagger(t_E,x)},$$

Kähler potential:

$$\mathcal{K} = \log\langle\lambda|\lambda\rangle = \log Z[\tilde{\lambda}], \quad \tilde{\lambda} = \lambda(t_E)\theta(-t_E) + \lambda^*(-t_E)\theta(t_E)$$



Quantum Kähler structure in QFT

Kähler form:

$$\Omega_\lambda = i\int_{t_E>0} dx \int_{t_E<0} dy \big[\langle O^\dagger(x)O(y)\rangle_{\tilde\lambda} - \langle O^\dagger(x)\rangle_{\tilde\lambda}\langle O(y)\rangle_{\tilde\lambda}\big] D\lambda^*(x)\wedge D\lambda(y),$$

Use instead matrix elements on small field variations

$$v = \int dx \Big[\delta\lambda(x)\frac{\delta}{\delta\lambda(x)} + \delta\lambda^*(x)\frac{\delta}{\delta\lambda^*(x)}\Big], \quad D\lambda(x)\Big[\frac{\delta}{\delta\lambda(y)}\Big] = \delta(x-y).$$

$$\begin{aligned}\Omega_\lambda(v_1, v_2) &= i(\delta_1\delta_2^* - \delta_2\delta_1^*)\log Z[\tilde\lambda] \\ &= i\int_{t_E>0} dx \int_{t_E<0} dy G^c_{\tilde\lambda}(x,y)\Big[\delta\lambda_1^*(x)\delta\lambda_2(y) - \delta\lambda_1(y)\delta\lambda_2^*(x)\Big],\end{aligned}$$

with

$$G^c_{\tilde\lambda}(x,y) = \langle O^\dagger(x)O(y)\rangle_{\tilde\lambda} - \langle O^\dagger(x)\rangle_{\tilde\lambda}\langle O(y)\rangle_{\tilde\lambda}.$$

Quantum Kähler structure in QFT

Another natural way of writing this:

$$\Omega_\lambda(v_1, v_2) = i \int_{t_E>0} dx \Big(\delta\lambda_1^* \delta_2 \langle O \rangle - \delta\lambda_2^* \delta_1 \langle O \rangle \Big).$$

Sources and vevs are canonically conjugate.

Wald's formalism

Classical mechanics	Classical field theory (including GR)
$S = \int dt L(q, \dot{q})$	$S = \int d^d x \mathcal{L}(\phi, \partial_\mu \phi)$
$\delta L = -(\mathsf{EOM})\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)$ $\alpha = p dq$ $\alpha(\delta q \partial_q + \delta p \partial_p) = p \delta q$	$\delta(\mathcal{L} d^d x) = -E_\phi \delta\phi d^d x + d\theta(\phi, \delta\phi)$
$\Omega = d\alpha = dp \wedge dq$ $\Omega(v_1, v_2) = \delta_1 p \delta_2 q - \delta_2 p \delta_1 q$	$\omega(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\theta(\phi, \delta_2\phi) - \delta_2\theta(\phi, \delta_1\phi)$ $\Omega(v_1, v_2) = \int_\Sigma \omega(\phi, \delta_1\phi, \delta_2\phi)$
$dH(.) = \omega(., X_H)$	$\delta H = \int_\Sigma \omega(\phi, \delta\phi, \mathcal{L}_t \phi)$

Quantum Kähler structure for holographic CFTs

In Holography:

$$\mathcal{K} = \log Z[\tilde{\lambda}] = -S^{\text{on-shell}}_{\text{grav}}[\tilde{\lambda}].$$

Varrying only the boundary condition:

$$\begin{aligned}\tilde{\delta} S^{\text{on-shell}}_{\text{grav}}[\tilde{\lambda}] &= \int_X d\theta(\phi, \delta\phi) \\ &= \int_{\partial X = S^d} \theta(\tilde{\lambda}, \delta\tilde{\lambda}).\end{aligned}$$

Because linearity of θ in the variation, if we do a strictly (anti)holomorphic variation, the integral localizes on the (upper)lower hemisphere:

$$\delta_1 S = \int_{t_E<0} \theta(\tilde{\lambda}, \delta\tilde{\lambda}_1), \quad \delta_1^* S = \int_{t_E>0} \theta(\tilde{\lambda}, \delta\tilde{\lambda}_1).$$

Quantum Kähler structure for holographic CFTs

We obtain the Kähler form

$$\begin{aligned}
\Omega_\lambda(v_1, v_2) &= i(\delta_1\delta_2^* - \delta_2\delta_1^*) S^{\text{on-shell}}_{\text{grav}}[\tilde{\lambda}] \\
&= i([\delta_1 + \delta_1^*]\delta_2^* - [\delta_2 + \delta_2^*]\delta_1^*) S^{\text{on-shell}}_{\text{grav}}[\tilde{\lambda}] \\
&= i(\tilde{\delta}_1\delta_2^* - \tilde{\delta}_2\delta_1^*) S^{\text{on-shell}}_{\text{grav}}[\tilde{\lambda}] \\
&= i\tilde{\delta}_1 \int_{t_E>0} \theta(\tilde{\lambda}, \delta\tilde{\lambda}_2) - \tilde{\delta}_2 \int_{t_E>0} \theta(\tilde{\lambda}, \delta\tilde{\lambda}_1) \\
&= i \int_{(\partial X)^+} \omega_{\text{bulk}}(\phi, \delta_1\phi, \delta_2\phi).
\end{aligned}$$

with $\phi|_{\partial X} = \tilde{\lambda}$.

Quantum Kähler structure for holographic CFTs

We obtain the Kähler form

$$\Omega_\lambda(v_1, v_2) = i \int_{(\partial X)^+} \omega_{\text{bulk}}(\phi, \delta_1\phi, \delta_2\phi).$$

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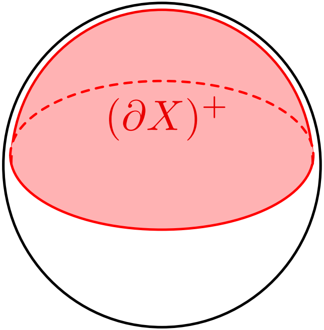


Quantum Kähler structure for holographic CFTs

We can push it to the bulk

$$0 = (\delta_2 E_\phi \delta\phi_1 - \delta_1 E_\phi \delta\phi_2) d^d x + d\omega_{\text{bulk}}(\phi, \delta_1\phi, \delta_2\phi),$$

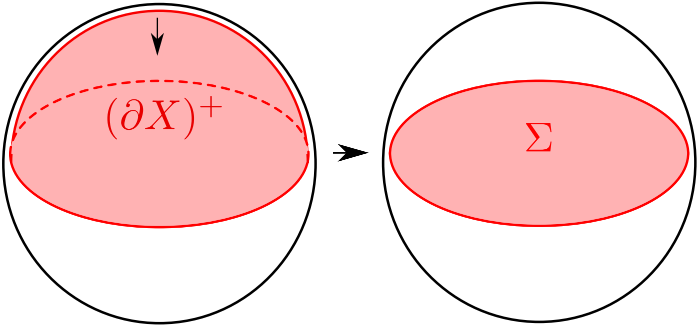
On-shell bulk variations satisfying boundary conditions:

$$\begin{aligned}\delta\phi(Y) &= \int_{\partial X} dy G_E(Y, y)\delta\tilde{\lambda} \\ &= \int_{(\partial X)^+} dy G_E(Y, t_E)\delta\lambda^*(-t_E) + \int_{(\partial X)^-} dy G_E(Y, t_E)\delta\lambda(t_E),\end{aligned}$$

Quantum Kähler structure for holographic CFTs

Therefore,

$$i\int_{(\partial X)^+} dx\Big(\delta\lambda_1^*\delta_2\langle O\rangle - \delta\lambda_2^*\delta_1\langle O\rangle\Big) = i\int_\Sigma \omega_{\text{bulk}}(\phi,\delta_1\phi,\delta_2\phi).$$



Lorentzian continuation

Where to push it?
For $\lambda = \lambda^*$ there is a Z_2 symmetric slice, $t_E = 0$.

By construction

$$\phi \in \mathbb{R}, \quad \pi = 0$$

or

$$h_{ij} \in \mathbb{R}, \quad K_{ij} = 0$$

Does not change under continuation to Lorentzian! Because of Z_2 symmetry, the variations:

$$\delta\phi(Y) = \int_{(\partial X)^+} dy G_E(Y, t_E)(\delta\lambda^*(-t_E) + \delta\lambda(-t_E)),$$
$$\delta\pi(Y) = \int_{(\partial X)^+} dy [i\partial_n G_E(Y, t_E)](\delta\lambda^*(-t_E) - \delta\lambda(-t_E)).$$

Real Lorentzian initial data variations $\Leftrightarrow$ $\delta\lambda^*$ is the conjugate of $\delta\lambda$.

Lorentzian continuation

Where to push it when $\lambda \neq \lambda^*$?

Less clear... Assume the $Z_2 + C$ symmetry extends to the bulk. Look for $Z_2 + C$ symmetric surfaces on which

$$\phi \in \mathbb{R}, \quad \pi \in i\mathbb{R}$$

or

$$h_{ij} \in \mathbb{R}, \quad K_{ij} = i\mathbb{R}$$

Good Lorentzian initial data. Lorentzian bulk variations are real provided bulk-to-boundary propagator respects the $Z_2 + C$ symmetry

$$G_E(Y, t_E) = G_E^*(T(Y), -t_E)$$

Complex structure and Kähler metric

Comes from the boundary inner product

$$J = i \int_{t_E<0} dx \Big(D\lambda(x) \frac{\partial}{\partial \lambda(x)} - D\lambda^*(x) \frac{\partial}{\partial \lambda^*(x)} \Big).$$

Complicated in the bulk. However, the Kähler norm of a variation

$$\begin{aligned} g(v_1, v_1) &= \Omega(v_1, J(v_1)) \\ &= \int_\Sigma \omega_{\text{bulk}} \Big(\phi, P_+ \delta\phi_1, (P_+\delta\phi_1)^* \Big), \end{aligned}$$

$$P_+ \delta\phi = \int_{(\partial X)^-} G_E(Y, y) \delta\lambda(y),$$

Klein-Gordon norm. Suggests: positive energy modes $\Leftrightarrow$ negative Euclidean times, see also [Marolf,Parrikar,Rabideau,Rad,Raamsdonk]

Special variations 1: time translation

Consider

$$|\lambda, \tau\rangle = e^{-\tau H}|\lambda\rangle, \quad \tau = it,$$

Regard τ as another complex source. Boundary symplectic form

$$\begin{aligned}\Omega(v, v_{\Delta t}) &= (i\Delta t \partial_\tau \delta_{\lambda^*} - \delta_\lambda (i\Delta t)^* \partial_{\tau^*}) \log\langle\lambda, \tau|\lambda, \tau\rangle \\ &= i\Delta t(\delta_\lambda + \delta_{\lambda^*})\frac{\langle\lambda|H|\lambda\rangle}{\langle\lambda|\lambda\rangle}.\end{aligned}$$

Equating with symplectic form

$$\delta\langle H\rangle = \int_\Sigma \omega_{\text{bulk}}(\phi, \delta\phi, \mathcal{L}_\xi \phi)$$

Special variations 2: extremal volume

In Einstein gravity, the symplectic form reads as

$$\Omega(v_1, v_2) = i \int_\Sigma \left(\delta_1 h_{ab} \delta_2 p^{ab} - \delta_2 h_{ab} \delta_1 p^{ab}\right)$$

$$p_{ab} = \sqrt{h}(K_{ab} - h_{ab} K), \quad K_{ab} = -\frac{1}{2}(\nabla_a n_b + \nabla_b n_a),$$

Consider the Weyl-like transformation [Jacobson]

$$\delta_w h_{ab} = 0, \quad \delta_w K_{ab} = \alpha h_{ab},$$

Leads to

$$\begin{aligned} \Omega(v_w, v_2) &= i\alpha(d-1) \int_\Sigma \sqrt{h} h^{ab} \delta_2 h_{ab} \\ &= 2i\alpha(d-1)\delta_2 V, \end{aligned}$$

Volume is the “conserved charge" conjugate to Weyl transformation.

Special variations 2: extremal volume

The initial data variation

$$\delta_w h_{ab} = 0, \quad \delta_w K_{ab} = \alpha h_{ab},$$

needs to satisfy the (linearized) momentum and Hamiltonian constraints to have a variation $(\delta_w \gamma_{ij}, \delta_w \gamma^*_{ij})$ of the boundary metric inducing it.

$$\delta_w(\nabla^j K_{jk} - \nabla_k K^j_j) = \alpha(\nabla^j h_{jk} - \nabla_k h^j_j) = 0$$
$$\delta_w\big(K_{ij}K^{ij} - (K)^2 - R_d - d(d-1)\big) = 2\alpha(d-1)K$$

Requires $K = K^i_i = 0$, i.e. the slice to be *extremal*.

Special variations 2: extremal volume

So what is $(\delta_w \gamma_{ij}, \delta_w \gamma^*_{ij})$**?**
We need this to give a completely boundary prescription. One can perturb AdS_3

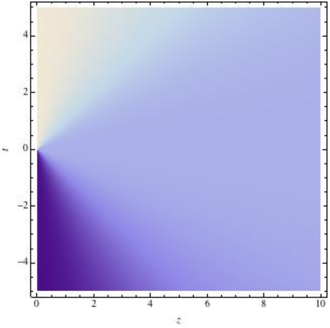
$$
\begin{aligned}
ds^2 = &\frac{dz^2 + dx^2 + dt^2}{z^2} \\
&+ \alpha \frac{t}{z^2(t^2+z^2)^{3/2}} \left[dz^2 z^2 + dx^2(t^2+z^2) + t^2 dt^2 + 2tz dt dz \right]
\end{aligned}
$$

- Solution of EOM up to order α^2
- Has $\delta h_{ij} = 0$, $\delta K_{ij} = \alpha h_{ij}$ for the $t = 0$ surface

Special variations 2: extremal volume

So what is $(\delta_w \gamma_{ij}, \delta_w \gamma^*_{ij})$?

- Boundary induced metric: $\tilde{\gamma}_{ij} = \delta_{ij}(1 + \alpha \mathrm{sign} t)$, Brown-York stress tensor consistent with this.
- Natural if $\alpha = ia$ is actually imaginary,
  $\delta_w \gamma_{ij} = -ia\gamma_{ij}, \quad \delta_w \gamma^*_{ij} = ia\gamma^*_{ij}$
- Discontinuity in boundary metric is smoothed out in the bulk. Also, no FG gauge. Similar to the story for conical singularities in the boundary [Lewkowycz-Maldacena,Camps]



Special variations 2: extremal volume

Brown-York stress tensor (or obtain it from the deformed boundary metric via $\langle T^i_i(x) T_{kl}(y)\rangle = c[\partial_k\partial_l - \delta_{kl}\nabla^2]\delta^2(x-y)$):

$$T_{tt} = T_{tx} = 0, \quad T_{xx} = -\alpha\delta'(t) \approx \frac{1}{\epsilon}\big[\delta(t+\epsilon) - \delta(t-\epsilon)\big]$$

(this is roughly how the bulk regulates it)
Therefore, the boundary symplectic form gives

$$\begin{aligned}\Omega(v, v_w) &\sim \int_{(\partial X)^+} \delta\tilde{\gamma}^{xx}\delta_w\langle T_{xx}\rangle - \int_{(\partial X)^-} \delta\tilde{\gamma}^{xx}\delta_w\langle T_{xx}\rangle^* \\ &= \int dx \Big[\frac{1}{\epsilon}\big[\delta\tilde{\gamma}^{xx}(0^+, x) + \delta\tilde{\gamma}^{xx}(0^-, x)\big] \\ &\quad + \big[\partial_t\delta\tilde{\gamma}^{xx}(0^+, x) - \partial_t\delta\tilde{\gamma}^{xx}(0^-, x)\big] + O(\epsilon)\Big]\end{aligned}$$

- Contains both a divergent and a finite piece
- Agrees with doing the calculation for the variation of the volume under $\delta\gamma_{ij}$ around AdS_3

What's left

- Move away from the vacuum. Natural guess: $\delta_w \gamma_{ij} = -ia\gamma_{ij}, \quad \delta_w \gamma^*_{ij} = ia\gamma^*_{ij}$. Predicts that variations of the volume are related to variations of integrals of the trace anomaly. Can only work in even d. We hope to prove it for $d = 2, 4$ and Z_2 symmetry, match divergences in other cases.
- Other special variations?
  - Picking up topological invariants on the bulk slice?
  - Regard modular time as a complex source $\rightarrow$ boundary symplectic form gives the relative entropy $\rightarrow$ connection to quasilocal energy[Lashkari,Raamsdonk]?
- Non-Z_2 symmetric states? TFD?

Questions?